UNIT – IV Complex integration

Basic concepts:

Connected region:

A region R is said to be connected when two points of it are connected by a curve which entirely lies inside the region.

Simply connected:

Any closed curve which lies in the region R, can be shrink to a point without learning R is called simply connected region.

Multiply connected region:

A region which is not simply connected is called multiply connected.

Cauchy's integral (Fundamental) theorem:

Statement:

If f(z) is analytic at every point of the region R bounded by a closed curve C and if f(z) is continuous at all points inside and on closed curve C, then $\int_C f(z)dz = 0$.

Cauchy's integral formula:

If f(z) is analytic inside and on a closed curve C of a simply connected region R and if a is any point within C, then f(a)=$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$.

Cauchy's integral formula for higher derivative:

$$f^n(a) = \frac{ni}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

Taylor's series

A function $f(z)$, analytic inside circle c with centre at $z = a$, can be expanded in the form $f(z) = f(a) + (z-a)\frac{f'(a)}{1!} + \frac{(z-a)^2}{2!}f''(a) + \cdots\cdots$ which is convergent at every point inside c.

Laurent's series

Let C_1, C_2 be two concentric circles $|z-a| = R_1$ **and** $|z-a| = R_2$ where $R_1 \prec R_2$. Let $f(z)$ be analytic on C_1 **and** C_2 and in the annular region R between them, then for any point z in R $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=0}^{\infty} b_n (z-a)^{-n}$ where

$$a_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^{1+n}}dz \text{ and } b_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^{1-n}}dz$$

Standard expansions:

1) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots\cdots \quad = \sum_{n=0}^{\infty}(-1)^n x^n \qquad if\,|x| < 1$

2) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots\cdots \quad = \sum_{n=0}^{\infty} x^n \qquad if\,|x| < 1$

3) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots\cdots \quad = \sum_{n=0}^{\infty}(-1)^n (n+1)x^n \qquad if\,|x| < 1$

4) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots\cdots \quad = \sum_{n=0}^{\infty}(n+1)x^n \qquad if\,|x| < 1$

5) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots \quad = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

6) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$

7) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots$

Evaluation of Residues

Case(i): if z= a is a simple pole (pole of order1) the residue of f(z) at z=a is given By $\lim_{z \to a}(z-a)f(z)$

Case(ii): if z = a is pole of order m then Residue of f(z) at z = a is given by

$$\frac{1}{(m-1)!}\lim_{z \to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

Case(iii): if f(z) = $\frac{P(x)}{Q(x)}$ has a simple pole at z = a then residue at z = a is given By $\frac{P(a)}{Q(a)}$ provide $Q^1(a) < 0$.

Resf(z) at z = a = coefficient of $\frac{1}{z-a}$ in Laurent series.

Cauchy's Residue theorem:

Statement:

If f(z) is analytic within and on a simple closed curve C except for a finite number of singularities $z_1, z_2, \cdots\cdots z_n$ lies inside C and if $R_1, R_2, \cdots\cdots R_n$ are the residues of f(z) at these singularities respectively ,then

$$\int_C f(z)dz = 2\pi i\left[R_1 + R_2 + \cdots\cdots + R_n\right]$$

Contour Integration:

Type I: $\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta \to (1)$

Put $z = e^{i\theta} \Rightarrow \cos\theta = \left(\frac{z^2+1}{2z}\right)$ and $\sin\theta = \left(\frac{z^2-1}{2iz}\right)$ and C is the unit circle

$|z| = 1, d\theta = \frac{dz}{iz}$ reduce type (1) to the form $\int_C f(z)dz$ which can be evaluated by Cauchy's residue theorem.

Type II: $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$

Consider $\int_C F(z)dz$ where $F(z) = \frac{P(z)}{Q(z)}$ and C is the closed contour consisting of the line segment

(-R,R) together with upper semi circle $|z| = R$

Cauchy's lemma:

If f(z) is continuous function such that $|zf(z)| \to 0$ uniformly as $|z| = R \to \infty$ then $\int_C f(z)dz \to 0$.

Type III: $\int_{-\infty}^{\infty} f(x)\sin mx\, dx \;\; (or) \;\; \int_{-\infty}^{\infty} f(x)\cos mx\, dx$

Jordan's lemma:

If f(z) is continuous function such that $|f(z)| \to 0$ is uniformly as $|z| = R \to \infty$ then $\int_C e^{imz} f(z)dz \to 0$

Evaluate the following by using Cauchy's Integral Theorem

1) $\int_C \frac{z^2 - z + 1}{z - 1} dz$ where C is $|z| = \frac{1}{2}$

Answer : 0

2) $\int_C \frac{z^2 + 1}{z^2 - 1} dz$ where C is $|z - 1| = 1$

Answer : 2π i

3) $\int_C \frac{4 - 3z}{z(z-1)(z-2)} dz$ where C is $|z| = \frac{3}{2}$

Answer : 2π i

4) $\int_C \frac{e^{2z}}{(z+1)^4} dz$ where C is $|z| = 2$

Answer : $\frac{8\pi i}{3e^2}$

5) $\int_C \frac{z}{(z-1)^2 (z+2)} dz$ where C is $|z| = \frac{3}{2}$

Answer : $\frac{4\pi i}{7}$

6) $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is $|z| = 3$

Answer : 4π i

7) $\int_C \frac{dz}{(z^2 + 4)}$ where C is $|z - i| = 2$

Answer : $\frac{\pi}{16}$

8) $\int_C \frac{z + 1}{z^2 + 2z + 4} dz$ where C is $|z + 1 + i| = 2$

Answer : π i

9) If $f(a)=\int_C \frac{4z^2+z+5}{(z-a)}dz$ where C is the ellipse $9x^2+4y^2=36$. Find the values of $f(1), f(i), f'(-1)$ **and** $f''(-i)$.
Answer : $20\pi i, 2\pi i(1+i), 14\pi i$ **and** $16\pi i$

10) If $f(a)=\int_c \frac{3z^2+7z+1}{z-a}dz$ where c is $|z| = 2$. Find $f(3),\ f(i-1),\ f'(1-i)$ **and** $f''(1-i)$
Answer : $0, 2\pi i(8-13i), 2\pi i(13-6i), 12\pi i$

11) Evaluate $\int_c \frac{e^z}{z(1-z)^3}dz$ if
(i) 0 lies inside c and 1 lies outside c
(ii) 1 lies inside c and 0 lies outside c
(iii) both 0 and 1 lies inside c.

Problems on Taylor's and Laurent's Series

1) Expand f(z) = sinz as Taylor's series about $z=\frac{\pi}{4}$

2) Expand $f(Z)=\frac{e^{2z}}{(z-1)^3}$ about z=1 using Taylor's series.

3) Expand $f(Z)=\frac{z^2-1}{(z+2)(z+3)}$ in Laurent's series

if (i) $|z|<2$ (ii) $2<|z|>3$ (iii) $|z|>3$

4) Find the Laurent's series of $f(z)=\frac{7z-2}{z(z-2)(z+1)}$ valid in the region
(i) $1<|z+1|<3$ (ii) $|z+1|>3$

5) Find the Laurent's series of $f(z)=\frac{z}{(z^2+1)(z^2+4)}$ valid in the region
(i) $1<|z|<2$ (ii) $|z|>2$

Problems on Cauchy's Residue Theorem

1) Determine Residue of the following at each of its poles
   a. (i) f(z) = cotz
   b. ii) f(z) = tanz
   Answer :1, -1

2) Find the Residue of f(z) = $z^2 \sin\left(\frac{1}{z}\right)$ at z = 0

3) Using Cauchy's Residue theorem , evaluate $\int_C \frac{3z^2+2}{(z-1)(z^2+9)}dz$ Where C is $|z-2|=2$

   Answer : πi

4) Evaluate $\int_C \frac{dz}{\sin z}$ where C is $|z|=4$ using Cauchy's residue theorem.

   Answer : $2\pi i$

5) Evaluate $\int_C \frac{z\sec z}{(1-z^2)}dz$ where C is the ellipse $4x^2+9y^2=9$

   Answer : $-2\pi i \sec 1$

6) Evaluate $\int_C \frac{z^2}{z^4-1}dz$ where C is $|z|=2$

   Answer :0

Problems on Contour Integration

(1) $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ where a>b>0

   Answer : $\frac{2\pi}{\sqrt{a^2-b^2}}$

2) $\int_0^{2\pi} \frac{d\theta}{1-2a\sin\theta+a^2}$ 0<a<1

   Answer : $\frac{2\pi}{1-a^2}$

3) Prove that $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$

Answer : $\frac{5\pi}{12}$

4) Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx$

Answer : $\frac{\pi}{a+b}$

5) Prove that $\int_{0}^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}$, m>0

6) Prove that $\int_{0}^{\infty} \frac{x \sin x}{x^2 + a^2} dx$

Answer : $\frac{\pi e^{-a}}{2}$