III. COMPLEX VARIABLE

3.1 The Complex Variable:

Cartesian Form : $z = x + iy$;

Polar form : $z = re^{i\theta}$

3.2 Function of a Complex Variable:

Cartesian form: $\omega = f(z) = u(x, y) + i\, v(x, y)$ Where x, y ε R and u (x, y), v (x, y) are real valued functions.

Polar form: $\omega = f(z) = u(r, \theta) + i\, v(r, \theta)$

3.3 Analytic function: A function f (z) is said to be Analytic in a region R of z-plane, if it is differentiable at every point of R.

3.3.1 Necessary Condition for function f (z) to be Analytic:

Cartesian form	Polar form
(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist	(i) $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ exist
(ii) f (z) to satisfy Cauchy-Riemann equation: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$	(ii) f (z) to satisfy Cauchy-Riemann equation: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} : \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

3.3.2 Sufficient Condition for function f (z) to be Analytic:

Cartesian form	Polar form
(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist	(i) $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ exist
(ii) They are Continuous	(ii) They are Continuous
(iii) f (z) to satisfy Cauchy-Riemann equation: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$	(iii) f (z) to satisfy Cauchy-Riemann equation: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} : \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

3.4 Harmonic function (or) potential function: A function which satisfies the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ then the function is called Harmonic function.

Example: $u(x,y)=x^3-3xy^2$ is a Harmonic Function

In polar coordinates: $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$

3.5 Properties of Analytic function:

1. The real and imaginary parts of an Analytic function ω = u + i v satisfy the Laplace equation in two dimension.
2. The real and imaginary parts of an Analytic function ω = u (r, θ) + i v (r, θ) satisfy the Laplace equation in polar coordinates.
3. If ω = u (x, y) + i v (x, y) is Analytic function, the curves of the family u (x, y) = a and v (x, y) = b, cut orthogonally, where a and b are varying constants.
4. If ω = u (r, θ) + i v (r, θ) is Analytic function, the curves of the family u (r, θ) = a and v (r, θ) = b, cut orthogonally, where a and b are varying constants.

3.6 CONSTRUCTION OF ANANYTIC FUNCTION

f (z) = u (x, y) + i v (x, y) / MILNE THOMSON METHOD

When its real part u (x, y) is given

$$f(z)=\int\left[u_x(z,0)-iu_y(z,0)\right]dz+c$$

When its Imaginary part v (x, y) is given

$$f(z)=\int\left[v_y(z,0)+iv_x(z,0)\right]dz+c$$

3.7 Conformal mapping:

A transformation that preserves angles between every pair of curves through a point, both in magnitude and sense is said to be *conformal* at that point.

3.8 Critical point:

A point at which $f'(z)=0$, is called a critical point of the transformation.

i.e. A point at which f (z) is not conformal mapping is called critical point.

Example:

The point z = 0 is a critical point of the transformation $w = z^2$.

3.9 Fixed point/Invariant point:

If the image of a point z under a transformation w = f(z) is itself, then the point is called a fixed point or a invariant point of the transformation.

Example:

$f(z)=\frac{3z-5}{z+1}$ Then the invariant points are 1 + 2i and 1 – 2i.

3.10 Cross-Ratio:

If $z_1, z_2, z_3 \;\&\; z_4$ are four complex numbers, then $\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$ is called the Cross-Ratio of the four points.

3.11 Bilinear Transformation (or) Linear fractional Transformation (or) Mobius Transformation.

The transformation $w=\frac{az+b}{cz+d}$ where a, b, c and d are complex constant and $ad-bc\neq 0$ is known as bilinear transformation.

Note: If $ad-bc=0$, every point of the z-plane is a critical point.

The Bilinear transformation which maps the points $(\omega_1, \omega_2, \omega_3, \omega_4)$ onto the points (z_1, z_2, z_3, z_4) respectively is

$$\frac{(\omega_1-\omega_2)(\omega_3-\omega_4)}{(\omega_1-\omega_4)(\omega_3-\omega_2)}=\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

Exercise

1. Find the analytic region of $f(z) = (x-y)^2 + 2i(x+y)$.

2. Examine whether e^z & Sin (z) are analytic.

3. Examine whether $|z|^2$ is analytic.

4. Examine whether z^2 & z^3 are analytic.

5. State the Properties of analytic function.

6. Give an example that both u & v are harmonic but f (z) is not analytic.

7. Prove that analytic function with (i) constant real part (ii) constant imaginary part is constant.

8. Prove that function with constant modulus is constant.

9. If f (z) is analytic prove that (i) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

(ii) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)Log|f(z)| = 0.$

10. Prove that $u(x, y) = x^3 - 3x^2y + 3x^2 - 3y^2 + 1$ is Harmonic and find v the harmonic conjugate, the Analytic function f (z).

11. Determine the analytic function f (z) if $v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$.

12. Verify $v = (x\cos y - y\sin y)e^x$ is harmonic. Construct analytic function.

13. Find f (z) if $u = \log\sqrt{x^2 + y^2}$

14. If $u + v = (x - y)(x^2 + 4xy + y^2)$ find the Analytic function f (z).

15. If $3u + 2v = y^2 - x^2 + 16xy$, find the Analytic function f (z).

16. Find the analytic function whose imaginary part is $e^{x^2-2y^2}\ sin(2xy)$.

17. In a two dimensional fluid flow if velocity potential $\varphi = 3x^2y - y^3$. Find Stream function ψ.

CONFORMAL MAPPING

18. Find the image of the y-axis under the transformation $w = z^2$.

19. Discuss the transformation (i) $w = \frac{1}{z}$. (ii) $w = z + \frac{k}{z}$ (iii) $w = z^2$

(iv) ω = Sinz

20. Draw image of square (0,0) (1,0) (1,1) &(0,1) under w =(1+i)z.

21. Find the image of $|z - 3i| = 3$ under the mapping $w = \frac{1}{z}$.

22. Find the image if $|z| = 2$ under w=z+3+2i

23. Find the Image of the triangular region in the z-plane bounded by the lines x = 0. y = 0, and x + y = 1 under the transformation $(i)\ w = 2z\ (ii)\ w = e^{i\pi/4} z$.

24. Find the Image of x=k under $w = \frac{1}{z}$.

25. Find the image of the infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $w = \frac{1}{z}$.

FIXED POINTS, CRITICAL POINTS

26. Find the critical points of $w = z^2$

27. Find the critical points of $w^2 = (z - \alpha)(z - \beta)$

28. Find the fixed points of $w = \frac{z+4}{z+1}$.

BILINEAR TRANSFORMATION

29. Find the bilinear transformation which maps the points (-1, 0, 1) to (0, i, 3i).

30. Find the bilinear transformation which maps the points (-1, i, 1) onto (1, i, -1).

31. Find the bilinear transformation which maps the points z = -1, 1, ∞ onto -i, -1, i

32. Find the bilinear transformation which maps the points (∞, i, 0) onto (0, i,∞).

PRACTICE PROBLEM

1. Find f (z) if $v = 3x^2y - y^3$

Ans: $f(z) = z^3 + c$

2. If $u = e^x[x\cos y - y\sin y]$, find f (z).

Ans: $f(z) = ze^z + c$

3. If $u(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$, find v, f (z).

Ans: $v = 3xy^2 + 4xy - x^3$, $f(z) = 2z^2 - iz^3 + c$

4. Prove that $v = \log(x^2 + y^2)$ is harmonic. Find the Real part of analytic function with this function v as its Imaginary part.

Ans: 2 $\tan^{-1} y/x$ +c

5. If $u - v = e^x(\cos y - \sin y)$ find f (z).

Ans: $f(z) = e^z + c$

6. If $2u + v = e^{2x}[(2x + y)\cos 2y + (x - 2y)\sin 2y]$.Find f (z).

Ans: $f(z) = ze^{2z} + c$

7. In a 2-dimensional flow, the stream function is $\psi = \tan^{-1} y/x$. Find the velocity potential φ. **Ans**: $\Phi = \frac{1}{2}log(x^2 + y^2)$

8. Find the image of $|z + 2i| = 2$ under $w = \frac{1}{z}$. **Ans:** 4y = 1

9. Find the image in the ω plane of the region of the z plane bounded by the straight lines x = 1, y = 1 and x + y =1 under the transformation $w = z^2$.

Ans: $v^2 = -4(u-1)$, $v^2 = 4(u+1)$, $u^2 = -2(v - \frac{1}{2})$.

10. Find the critical points of $w = z^4 - 4z$ **Ans:** z = 1, $\frac{-1 \pm i\sqrt{3}}{2}$

11. Find the fixed points of $w = \frac{z-1-i}{z+2}$ **Ans:** $z = \frac{-1 \pm i\sqrt{3+4i}}{2}$

12. Find the bilinear transformation which maps the points z = 1, i,-1 onto i, 0,-I

Ans: $\omega = \frac{-z+i}{z+i}$

13. Find the bilinear transformation which maps the points (-i, 0, i) onto the points (-1, i, 1) respectively

Ans: $\omega = \frac{i(1-z)}{z+1}$

14. Find the bilinear transformation which maps the points (1, i,-1) onto (0, 1,∞).

Ans: $\omega = \frac{-iz+i}{z+1}$