UNIT -II

APPLICATIONS OF LAPLACE TRANSFORMS

Recall

1. If $L[f(t)] = F(s)$ then

(i) $L\left[f'(t)\right] = s\ L[f(t)] - f(0)$

(ii) $L\left[f''(t)\right] = s^2\ L[f(t)] - s\,f(0) - f'(0)$

In General $L\left[f^n(t)\right] = s^n\ L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots\ldots - f^{n-1}(0)$

(iii) $L\left[\int_0^t f(t)dt\right] = \frac{1}{s} L[f(t)]$

2. (i) $\int_0^t f(u)\,g(t-u)\,du = f(t) * g(t)$

(ii) $L[f(t) * g(t)] = L[f(t)]\ L[g(t)]$

(or)

$L\left[\int_0^t f(u)\ g(t-u)\ du\right] = L[f(t)]\ L[g(t)]$

Solve the following Ordinary Differential equations using Laplace Transforms

1. $\frac{dy}{dt} - y = 1 - 2t$ given that $y(0) = -1$
2. $y'' + y = 2e^t$, $y(0) = 1$, $y'(0) = 2$
3. $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 8y = e^{2t}$, $y(0) = 2$, $y'(0) = -2$
4. $y'' + 4y' + 3y = \sin t$, $y(0) = y'(0) = 0$
5. $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = \cos t$, if $\frac{dx}{dt} = 0$ and $x = 2$ when $t = 0$
6. $y'' - 3y' + 2y = 4$, $y(0) = 2$, $y'(0) = 3$
7. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = te^{-t}$, $y(0) = 2$, $y'(0) = -2$
8. $y'' - 2y' + y = (t+1)^2$, given $y(0) = 4$ and $y'(0) = -2$
9. $y''' - 3y'' + 3y' - y = t^2e^t$, given $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$

Practice Problems

Solve the following Ordinary Differential equations using Laplace Transforms

1. $y'' + 5y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = -1$

Ans : $y = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}$

2. $y'' - 3y' + 2y = e^{2t}, \quad y(0) = -3, \quad y'(0) = 5$

Ans : $y = -10e^{t} + 7e^{2t} + te^{2t}$

3. $(D^2 - 2D + 1)y = e^{t}$, $y = 2$ and $Dy = -1$ at $t = 0$

Ans : $y = 2e^{t} - 3te^{-t} + \frac{t^2}{2}e^{t}$

4. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = 1, \quad y = 0, \quad \frac{dy}{dt} = 1$ when $t = 0$

Ans : $\frac{1}{8} - \frac{1}{8}e^{-2t}\cos 2t + \frac{3}{8}e^{-2t}\sin 2t$

5. $y'' + y = t^2 + 2t, \quad y(0) = 4, \quad y'(0) = -2$

Ans : $y = \frac{t^3}{3} + 2 + 2e^{-t}$

6. $y'' - 6y' + 9y = t^2e^{3t}$, given $y(0) = 2, \; y'(0) = 6$

Ans : $\frac{1}{12}t^4e^{3t} + 2e^{3t}$

7. $(D^2 + 2D + 5)y = e^{-t}\sin t, \; y(0) = 0, \; y'(0) = 1$

Ans: $\frac{e^{-2t}}{3}(\sin t + \sin 2t)$

Solve the following Integral equations

1. $y + \int_0^t y(t)dt = e^{-t}$

2. $y + \int_0^t y(t)dt = t^2 + 2t$

3. $\frac{dy}{dt} + 2y + \int_0^t y(t)dt = 2\cos t, \quad y(0) = 1$

4. $x + \int_0^t x(t)dt = \cos t + \sin t$

5. $y' + 4y + 5\int_0^t y(t)dt = e^{-t}, \quad y(0) = 0$

6. $y = 1 + \int_0^t y(u)\sin(t-u)\,du$

7. $f(t) = \cos t + \int_0^t e^{-u} f(t-u)\,du$

8. $y(t) = t + \int_0^t \sin u\, y(t-u)\,du$

9. $y = e^{-t} - 2\int_0^t y(u)\cos(t-u)\,du$

10. $y'(t) = t^2 + \int_0^t \cos u\; y(t-u)\,du, \qquad y(0) = 4$

Practice Problems
Solve the following Integral equations using Laplace Transforms

1. $\frac{dy}{dt} - 2y + \int_0^t y(t)dt = 0, \quad y(0) = 1$

Ans : $e^t(1+t)$

2. $y' + 3y + 2\int_0^t y(t)dt = t, \quad y(0) = 0$

Ans : $\frac{1}{2} - \frac{1}{2}e^{-\frac{3}{2}t}\cosh\left(\frac{t}{2}\right) - \frac{3}{2}e^{-\frac{3}{2}t}\sinh\left(\frac{t}{2}\right)$

3. $x + \int_0^t x(t)dt = 1 - e^{-t}$

Ans : te^{-t}

4. $y = 1 - e^t + \int_0^t \sin(t-u)\ y(u)\ du$

Ans : $2 + t + \frac{t^2}{2} - 2e^t$

5. $y = 1 + 2\int_0^t e^{-2u}\ y(t-u)\,du$

Ans : $1 + 2t$

Solve the following Simultaneous Differential equations using Laplace Transforms

1. $\frac{dx}{dt} + y = \sin t$

   $\frac{dy}{dt} + x = \cos t,\ given \quad x(0) = 2,\ y(0) = 0$

2. $\frac{dx}{dt} + 3x - 2y = 1,\ \frac{dy}{dt} - 2x + 3y = e^t,\quad x(0) = 0,\ y(0) = 0$

3. $\frac{dx}{dt} + 2x - y = -6t,\quad \frac{dy}{dt} - 2x + y = -30t,$ given that $x = 2,\ y = 3$ when $t = 0$

4. $\frac{dx}{dt} + \frac{dy}{dt} + x - y = e^{-t}$

   $\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = e^t,$ Where $x(0) = 1,\ y(0) = 0$

5. $\frac{d^2x}{dt^2} + y = -5\cos 2t$

   $\frac{d^2y}{dt^2} + x = 5\cos 2t,$ given $x(0) = 0, x'(0) = 0, y(0) = 0, y'(0) = 0$

Practice Problems
Solve the following Simultaneous Differential equations using Laplace Transforms

1. $\frac{dy}{dt} + ay = x,\ \frac{dx}{dt} + ax = y,$ given that $x = 0,\ y = 1$ when $t = 0$

   Ans : $x = e^{-at}\sinh t,\quad y = e^{-at}\cosh t$

2. $\frac{dx}{dt} - 2x + 3y = 0,\ \frac{dy}{dt} - y + 2x = 0,\quad x(0) = 8,\ y(0) = 3$

   Ans : $x = 5e^{-t} + 3e^{4t},\quad y = 5e^{-t} - 2e^{4t}$

3. $(D+1)x + Dy = e^{-t}$

   $(D+2)x + (2D+2)y = 0,$ Where $x(0) = -1,\ y(0) = 1$

   Ans : $x = \frac{1}{2}e^t(\cos t + \sin t) - \frac{1}{2}\cos 2t,\quad y = -e^t(\cos t - \sin t) - \sin 2t$

4. $\frac{dx}{dt} - \frac{dy}{dt} - 2x + 2y = 1 - 2t,\quad \frac{d^2x}{dt^2} + 2\frac{dy}{dt} + x = 0,$ with $x = 0,\ y = 0,\ \frac{dx}{dt} = 0$ when $t = 0$

   Ans : $x = 2(1 - e^{-t} - e^{-t}t),\quad y = 2 - t - 2e^{-t} - 2te^{-t}$